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INDEPENDENCE, CLIQUE SIZE AND MAXIMUM DEGREE

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Dedicated to Paul Erdős on his seventieth birthday

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It was shown before that if G is a graph of maximum degree p containing no cliques of the size q then the independence ratio is greater than or equal to $\frac{2}{p+q}$. We shall discuss here some extreme cases of this inequality.

0. Throughout this paper G will be a finite simple n-vertex graph of maximum degree p, of clique size $q-1$ and independence α.

We proved in [4] that if $p \geqq q$, then

$$\frac{\alpha}{n} \geqq \frac{2}{p+q}. \tag{1}$$

We shall discuss here some cases in which (1) becomes an equality.

Theorem. *If* $q \leqq p$ *then* $\frac{\alpha}{n} = \frac{2}{p+q}$ *implies that* $3q-2p \leqq 5$. *For all natural numbers* p_1 *and* q_1 *such that* $3q_1 - 2p_1 = 5$ *there is a unique connected graph* G *with* $p = p_1$, $q = q_1$ and $\frac{\alpha}{n} = \frac{2}{p+q}$.

1. Since graphs satisfying $\frac{\alpha}{n} = \frac{2}{p+q}$ must have a large chromatic number and a high degree of symmetry (see the proof of the Theorem) we shall call them Butterflies.

A few examples come from Ramsey theory. If G is a regular triangle-free graph in which independence is equal to the degree then its complement is a Butterfly. Thus a few known regular $R(3, n)$ Ramsey graphs provide simplest examples. So do Greenwood—Gleason and Chvátal graphs.

Ramsey's Theorem implies also that for every p and q there are at most finitely many Butterflies. Conversely (1) gives a bound for Ramsey numbers, [4].

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The strong direct product of a complete graph and a Butterfly is again a Butterfly.

Strong direct products of pentagon and complete graphs proves the existence part of our theorem. These graphs were described in [4] in what now seems to be a rather complicated manner.

The same graphs are among Catlin's counterexamples to Hajós conjecture, [2] and Catlin told me that they were also (probably first) used by Gallai. $C_2 \times C_5$ is one of two critical Butterflies of [1].

Erdős and I proved that for almost all graphs the Hajós ratio is much greater than expected, [3]. Our proof is based on random Ramsey graphs. Perhaps some new Butterflies will be relevant to some open cases of Hajós or some related conjectures like, for example, that of Hadwiger, or that of [3].

2. If S is a subset of G then $\alpha_k(S)=\alpha_k$ will denote the number of all vertices adjacent to exactly k vertices of S. We will refer to these vertices as α_k-vertices. $G(x)$ will be the set of vertices adjacent to x and $|S|$ will stand for the number of elements of S. We shall sometimes refer to elements of $G(x)\cap S$ as roots of x.

We shall prove now that if G is a Butterfly then $3q-2p\leqq 5$.

If $S=I$ is a maximum independent set then we have

(2) $\quad \alpha_1+\alpha_2+\ldots+\alpha_p = n-\alpha$

(3) $\quad \alpha_1+2\alpha_2+\ldots+p\alpha_p \leqq p\alpha$

(4) for each $a\in I$ the set of all α_1-vertices adjacent to the vertex a forms a clique with at most $q-2$ elements, in particular $\alpha_1 \leqq \alpha(q-2)$.

Subtracting the inequality (3) from (2) multiplied by 2 and combining it with (4) we have

$$(p+q)\alpha \geqq 2n+\alpha_3+2\alpha_4+\ldots+(p-2)\alpha_p.$$

This is essentially the proof of (1), [4].

Now if G is a Butterfly then in all involved inequalities both sides must be equal. In particular

(5) for every maximum independent set we have $\alpha_3=\alpha_n=\ldots=\alpha_p=0$.

(6) The set of α_1-neighbors of any $a\in I$ is a clique with $q-2$ vertices.

Clearly every α_1 vertex can be exchanged in I for its root and a new set will be again maximum independent. The same can be done with every α_2 vertex a, since (6) implies that a is independent with one of the α_1-vertices adjacent to the roots of a. Thus every vertex of G is contained in a maximum independent set and hence, again the extreme case of (3) implies that

(7) G is regular of degree p.

(8) Let a be an α_2-vertex adjacent to a_1 and a_2 belonging to I. Then every α_1-vertex x adjacent to a is adjacent to either a_1 or a_2.

Indeed, if the unique vertex $b\in I$ to which x is adjacent were different from a_1

and a_2, we could replace in I the vertex b by x to obtain a new maximum independent set. But with respect to this new set a would be an α_3-vertex, contradicting (5).

(9) Let a, a_1, a_2 be the same vertices as in (8) and let P be the set of all α_1-vertices adjacent to a. Then $|P| \geqq q-3$.

Proof of (9). Let R be the set of α_1-vertices adjacent to a_1 or a_2. By (8) $P \subseteqq R$. We shall show that $R \setminus P$ is a clique. Indeed if two vertices x and y of $R \setminus P$ were independent then replacing in I the set $\{a_1, a_2\}$ by $\{a, x, y\}$ we would get a larger independent set. But this contradicts the choice of I. Thus $|R \setminus P| \leqq q-1$ and hence $|P| \geqq 2(q-2)-(q-1)=q-3$. ∎

(10) Let a be an α_2-vertex with roots a_1 and a_2, and let S be the set of α_1-vertices adjacent to a and to a_1. Then $|S| \leqq p-q+1$.

Proof of (10). One of α_1-vertices x adjacent to a_1 is not adjacent to a; else by (6) we would have a q-element clique consisting of a, a_1 and $(q-2)$ α_1-neighbors of a_1, (6). Replacing in I the set $\{a_1, a_2\}$ by $\{x, a\}$ we have a new maximum independent set. With respect to this new set all elements of S as well as a_1 are α_2-vertices. But since every element of a maximum independent set has $q-2$ α_1-neighbors therefore it must have $p-q+2$ α_2-neighbors. Thus $|S| \leqq p-q+1$. ∎

(11) If G is a Butterfly then $3q-2p \leqq 5$.

Proof of (11). Let P be the set defined in (9). By (9) and (10) we have

(12) $$q-3 \leqq |P| \leqq 2(p-q+1).$$

This proves (11). ∎

3. We shall prove now that *if G is a connected Butterfly with $3q-2p=5$ then $\alpha=2$.*

The equality in (12) implies that

(13) $$|P| = q-3,$$

and similarly

(14) $$|S| = p-q+1 = \frac{q-3}{2}, \quad (S \text{ was defined in (10)}).$$

(15) If a and b are α_2-vertices with disjoint sets of roots then a and b are independent.

Indeed, otherwise we could replace in I roots of b by b and an α_1-vertex independent with b. With respect to this new set a would be an α_3-vertex.

(16) If $q \geqq 5$ and b_1 and b_2 are α_2-vertices with roots $\{a_1, a_2\}$ and $\{a_2, a_3\}$, respectively then b_1 and b_2 are adjacent.

The assumption $q \geqq 5$ implies that there is an α_1-vertex x adjacent to a_2, but independent with both b_1 and b_2. Thus if $\{b_1, b_2\}$ were independent the set $(I \setminus \{a_1, a_2, a_3\}) \cup \{b_1, b_2, x\}$ would be maximum independent. But with respect to this set a_2 is an α_3-vertex.

Suppose that $|I| \geqq 3$. Since G is connected by (8) and (15) there are three $a_1, a_2, a_3 \in I$ and two α_2-vertices b_1 and b_2 such that b_1 is adjacent to a_1 and a_2, and b_2 is adjacent to a_2 and a_3.

Let Q be the set of α_1-neighbors of a_2 not adjacent to b_1 nor b_2. By (14), $|Q| \geqq q-2-2(p-q+1)=1$. Any element r of Q is adjacent to every α_1-vertex element of $G(a_1) \setminus G(b_2)$ and $G(a_3) \setminus G(b_2)$. By (13) and (14) these two sets have together $q-3$ elements.

But r is also adjacent to a_2 and $q-3$ α_1-neighbors of a_2. Thus its degree is at least $q-3+q-2 \leqq p = \frac{3q-5}{2}$, i.e., $q \leqq 5$, and since q is odd, $q > 3$ we have $p=q=5$.

Let b_3 denote another α_3-neighbor of a_3. By (16) b_2 is adjacent to b_1 and b_3. By (14) b_2 is adjacent to two α_1 vertices. But this means that the degree of b_2 is at least 6, which is a contradiction.

4. To show the *uniqueness* of connected Butterflies with $3q-2p=5$, we can assume now that $I=\{a_1, a_2\}$. Let A be the set of α_2-vertices of I and let $a \in A$. Furthermore, let A_i be the set of α_1-neighbors of a_i adjacent to a and B_i be the set of all α_1-neighbors of a_i not adjacent to a. Those are clearly all verticles of G. By (9) $B_1 \cup B_2$ is a clique. By (14), $|B_i| = q-2-(p-q+1) = \frac{q-1}{2}$. Since vertices of $B_1 \cup B_2$ are α_1-vertices by (7) this accounts for $(q-2)+\frac{q-1}{2}=p$ neighbors of every element of $B_1 \cup B_2$.

In particular every vertex of A is adjacent to a vertex of $A_1 \cup A_2$, and since by (13) we have that $|A_1 \cup A_2| = q-3$, (9) implies that every vertex of A is adjacent to every vertex of $A_1 \cup A_2$. Now every vertex of A must be adjacent to $p-(q-3)-2 = p-q+1$ vertices of A. But $|A| = p-q+2$ which means that A is complete.

This completely determines the adjacency relation for G and thus proves our theorem. ▮▮

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